Empirical Welfare Maximization with Constraints

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Abstract

When designing eligibility criteria for welfare programs, policymakers naturally want to target the individuals who will benefit the most. This paper proposes two new econometric approaches to selecting an optimal eligibility criterion when individuals' costs to the program are unknown and need to be estimated. One is designed to achieve the highest benefit possible while satisfying a budget constraint with high probability. The other is designed to optimally trade off the benefit and the cost from violating the budget constraint. The setting I consider extends the previous literature on Empirical Welfare Maximization by allowing for uncertainty in estimating the budget needed to implement the criterion, in addition to its benefit. Consequently, my approaches can be applied to settings with imperfect take-up or varying program needs. I apply these two approaches to derive an optimal budget-constrained Medicaid expansion in the US using estimates from the Oregon Health Insurance Experiment. Overall, I find that my approaches improve the existing approach both theoretically and empirically.

1 Introduction

When a welfare program induces varying benefits across individuals, and when resources are scarce, policymakers naturally want to prioritize eligibility to individuals who will benefit the most. Based on experimental data, cost-benefit analysis can inform policymakers on which subpopulations to prioritize, but these subpopulations might not align with any available eligibility criterion such as an income threshold. Kitagawa and Tetenov (2018) propose Empirical Welfare Maximization (EWM) that can directly select an eligibility criterion from a set of available criteria. For example, if available criteria take the form of income thresholds, EWM considers the problem of maximizing the expected benefits in the population

$$\max_{t \leq \bar{t}} E[benefit \cdot \mathbf{1}\{income \leq t\}]$$

and approximates the optimal threshold based on benefits estimated from experimental data. Recent papers have studied the statistical properties of the EWM approach, including Athey and Wager (2020), among others. Notably the average benefits under the eligibility criterion selected by EWM converges to the highest attainable benefits as the sample size grows.

In practice policymakers often face budget constraints, but only have imperfect information about whether a given eligibility criterion satisfies the budget constraint. First, there may be imperfect take-up: eligible individuals might not participate in the welfare program, resulting in zero cost to the government, e.g. Finkelstein and Notowidigdo (2019). Second, costs incurred by eligible individuals

*I am grateful to Alberto Abadie, Isaiah Andrews, Amy Finkelstein and Anna Mikusheva for their guidance and support. I thank Ben Deaner, Chishio Furukawa, Jasmin Fliegner, Jon Gruber, Yunan Ji, Joseph Kibe, Sylvia Klosin, Kevin Kainan Li, James Poterba, Emma Rackstraw, Jonathan Roth, Dan Sichel and Cory Smith for helpful discussions. I acknowledge generous support from the Jerry A. Hausman Graduate Dissertation Fellowship. This draft is a work in progress and comments are welcome; all errors are my own. Please check http://economics.mit.edu/files/20473 for the latest version.

who participate in the welfare program may vary considerably, largely driven by individuals' different needs but also many other factors, e.g. Finkelstein et al. (2017). Both considerations are hard to predict ex ante, implying that the potential cost of providing eligibility to any given individual is unknown at the time of designing the eligibility criterion. Unobservability of the potential cost requires estimation based on experimental data, contributing to uncertainty in the budget estimate of a given eligibility criterion. The existing EWM approach takes cost as known, so that in the aforementioned example the upper bound $\bar{t}$ for the income threshold that satisfies the budget constraint is known. Therefore, the existing approach is not able to address issues such as a potential cost overrun due to an underestimate of the budget.

In this paper I provide two new econometric approaches to selecting an optimal eligibility criterion when the budget needed to implement the eligibility criterion involves an unknown cost and need to be estimated. Both approaches take the form of a statistical rule, which selects an eligibility criterion based on the experimental data. The two statistical rules I propose augment the EWM literature in settings of unknown cost, each satisfying one of the two desirable properties I describe below. The first statistical rule I propose, the *mistake-controlling rule*, achieves the highest benefit possible while satisfying a budget constraint with high probability. Therefore, with high probability, the mistake-controlling rule selects only feasible eligibility criteria that satisfy the budget constraint. The second statistical rule I propose, the *trade-off rule*, can select infeasible eligibility criteria, but only if borrowing money and exceeding the budget constraint are not too costly to justify the marginal gain in benefit from violating the budget constraint. The choice between the two rules should align with policymakers' attitude toward the budget constraint. If policymakers are financially conservative, then the eligibility criterion selected by the mistake-controlling rule is more suitable. If policymakers want to reach as many individuals as they can, then the eligibility criterion selected by the trade-off rule is more suitable.

In this novel setting where the budget needed to implement an eligibility criterion involves an unknown cost, I evaluate statistical rules based on two properties: *asymptotic feasibility* and *asymptotic optimality*. A statistical rule is asymptotically feasible if given a large enough sample size, the statistical rule is very likely to select feasible eligibility criteria that satisfy the budget constraint. A statistical rule is asymptotically optimal if given a large enough sample size, the statistical rule is very likely to select eligibility criteria that achieve weakly higher benefits than any feasible criterion. A similar notion of asymptotic optimality has been studied by the existing EWM literature. The existing EWM statistical rule is proven to be asymptotically optimal in settings where cost is known. Departing from the existing EWM literature, I focus on settings with unknown cost. Correspondingly, I add asymptotic feasibility to asymptotic optimality as desirable properties for statistical rules. Ideally, the statistical rule should maintain good performance even when the data distribution is unknown. I therefore define *uniform asymptotic optimality* and *uniform asymptotic feasibility*, which impose *asymptotic optimality* and *asymptotic feasibility* uniformly over a class of reasonable data distributions, respectively. However, the setting with unknown cost can be too complicated for an arbitrary draw of the data to approximate, and I prove that it is impossible for any statistical rule to satisfy these two uniform properties simultaneously. Since achieving simultaneous uniform asymptotic feasibility and uniform asymptotic optimality is impossible, I instead aim to achieve one at a time.

The first statistical rule I propose, the mistake-controlling rule, is uniformly asymptotically feasible, meaning that given a large enough sample size, it is likely to select feasible eligibility criteria that satisfy the budget constraint regardless of the underlying data distribution. The second statistical rule I propose, the trade-off rule, can select infeasible eligibility criteria under any data distribution, but only to trade off the cost of exceeding the budget constraint against gains in the benefits. I prove this statistical rule is uniformly asymptotically optimal. These two alternative statistical rules I propose can also be applied to settings beyond budget-constrained welfare programs. For example, problems of maximizing the accuracy of any predictive algorithm while controlling the disparity in mistakes made across different ethnic groups have the same mathematical structure and fall under the framework of this paper.

To illustrate the statistical rules proposed in this paper, I consider an optimal budget-constrained Medicaid expansion. Medicaid is a government sponsored health insurance program intended for the low-income population in the United States. Based on the Oregon Health Insurance Experiment (OHIE) conducted in 2008, I derive Medicaid expansion criteria using the two statistical rules I propose. The current Medicaid expansion criterion in Oregon is based on a threshold for household income only. In my application, I examine whether health can be improved by allowing the income threshold to vary by number of children in the household, setting the budget constraint to the current level. The mistake-controlling rule selects an eligibility criterion that limits eligibility to be lower than the current level. This lower level occurs because Medicaid costs vary considerably across households, making it harder to verify whether households meet the budget constraint. In contrast, the trade-off rule selects an eligibility criterion that expands eligibility for many households above the current level, especially those with children. The higher level occurs because Medicaid in general improves health for these households, and the additional health benefit from violating the budget constraint exceeds the cost of doing so, assuming reasonable upper bound on the monetary value on the health benefit.

The rest of the paper proceeds as follows. Section 1.1 discusses related work in more detail. Section 2 formalizes the constrained optimization problem that policymakers aim to solve, where the budget constraint involves an unknown cost. It also proposes statistical rules that have good asymptotic properties in this setting. Section 3 conducts experiments to illustrate the asymptotic properties of the statistical rules I propose, and designs a more flexible Medicaid eligibility criterion for the low-income population in Oregon. Section 4 concludes.

1.1 Literature review

This paper is related to the traditional literature on cost-benefit analysis, e.g. Dhailiwal et al. (2013), and to the recent literature on EWM, e.g. Kitagawa and Tetenov (2018), Mbakop and Tabord-Meehan (2019), Rai (2019) and Athey and Wager (2020). More broadly, this paper contributes to a growing literature on statistical rules in econometrics, including Manski (2004), Dehejia (2005), Hirano and Porter (2009), Stoye (2009), Chamberlain (2011), Bhattacharya and Dupas (2012), Demirer et al. (2019) and Kasy and Sautmann (2020). The traditional cost-benefit analysis is only informative for whether this welfare program should be implemented with the fixed eligibility criterion as implemented in the RCT. EWM informs whether more flexible eligibility criteria than the one implemented in the RCT can improve welfare, an advantage over the traditional cost-benefit analysis. Kitagawa and Tetenov (2018) start with functional form restrictions on the class of available criteria, and then select the criterion with the highest estimated benefit (empirical welfare) based on an RCT sample. They prove the optimality of their statistical rule in the sense that its regret converges to zero at the minimax rate. Athey and Wager (2020) propose doubly-robust estimation of the average benefit, which leads to an optimal rule even with quasi-experimental data. Mbakop and Tabord-Meehan (2019) propose a Penalized Welfare Maximization assignment rule which relaxes restrictions of the criterion class. The theoretical contribution of this paper is to extend the literature by allowing both functional form restrictions on the available eligibility criteria and budget constraints with an unknown cost.

2 Statistical rules that account for budget constraints

2.1 The constrained optimization problem

I begin by setting up a general constrained optimization problem, which depends on the following random attributes of an individual:

$$A = (\Gamma, R, X) \in \mathcal{A} \subseteq \mathbb{R}^{2+p}. \tag{1}$$

Here Γ is her benefit from being eligible for the welfare program, R is her cost to the policymaker after she becomes eligible, and $X \in \mathcal{X} \subset \mathbb{R}^p$ denotes her p-dimensional observed characteristics. The individual belongs to a population that can be characterized by the joint distribution P on her random attributes A. The unknown distribution $P \in \mathcal{P}$ is from a class of distributions $\mathcal{P}$.

An eligibility criterion $g(X) \in \{0, 1\}$ determines the eligibility status for an individual with observed characteristics X, where 1 is eligibility and 0 is non-eligibility. Let $\mathcal{G}$ denote the class of criteria policymakers can choose from. The optimization problem is to find a criterion with maximal benefit while subject to a constraint on its cost:[1]

$$\max_{g \in \mathcal{G}} E_P[\Gamma \cdot g(X)] \; s.t. \; E_P[R \cdot g(X)] \leq k. \tag{2}$$

[1]Following Kitagawa and Tetenov (2018), I implicitly assume the maximizer exists in $\mathcal{G}$ with the notation in (2).

The benefit-cost attributes are unobserved and need to be estimated. Specifically, let (Y_1, Y_0) denote the potential outcomes that would have been observed if an individual were assigned with and without eligibility, respectively. The benefit from eligibility criterion is therefore defined as $\Gamma \coloneqq Y_1 - Y_0$. Note that maximizing benefit is equivalent to maximizing the outcomes (welfare) under the utilitarian social welfare function: $E_P[Y_1 \cdot g(X) + Y_0(1 - g(X))]$. Similarly, the cost R is unobserved before an individual receives her eligibility.

The appropriate expressions for their estimates depend on the type of observed data. Below I state the estimates formed based an RCT that randomly assigns the eligibility. The observed data from the RCT $\{A_i^*\}_{i=1}^n$ consists of i.i.d. observations $A_i^* = (Y_i, Z_i, D_i, X_i) \in \mathcal{A}^*$. The distribution of A_i^* is induced by the distribution of (Γ, R, X) as in the population, as well as the sampling design of the RCT. Here D_i is an indicator for being in the eligibility arm of the RCT, Y_i is the observed outcome and Z_i is the observed cost of providing eligibility to an individual participating in the RCT. The observed cost is mechanically zero if an individual is not randomized into the eligibility arm. The estimates for (Γ, R) are

$$\Gamma_i^* = \alpha(X_i, D_i) \cdot Y_i,\ R_i^* = \frac{D_i}{p(X_i)} \cdot Z_i, \tag{3}$$

where $\alpha(X_i, D_i) = \frac{D_i}{p(X_i)} - \frac{1-D_i}{1-p(X_i)}$ and $p(X_i)$ is the propensity score, the probability of receiving eligibility conditional on the observed characteristics. Since the sampling design of an RCT is known, the propensity score is a known function of the observed characteristics. For experiments where the the propensity score $p(X_i)$ needs to be estimated, such as in my application, I use the doubly-robust scores as estimates. I prove that the estimation errors are asymptotically unimportant in Appendix D.3.

To simplify the notation, I denote the expected benefit and the expected cost of criterion g under distribution P by the following population *welfare* function and *budget* function:

$$W(g; P) = E_P[\Gamma \cdot g(X_i)],\ B(g; P) = E_P[R \cdot g(X_i)]. \tag{4}$$

Given experimental data, using estimates for (Γ, R) described above, one can calculate the sample analogs to the welfare function and the budget function:

$$\widehat{W}_n(g) \coloneqq \frac{1}{n} \sum_i \Gamma_i^* \cdot g(X_i),\ \widehat{B}_n(g) \coloneqq \frac{1}{n} \sum_i R_i^* \cdot g(X_i). \tag{5}$$

Maximizing the benefit with respect to eligibility criterion is equivalent to maximizing the welfare, which is why I refer to $W(g; P)$ as the welfare function. The welfare function and the budget function are both deterministic functions from $\mathcal{G} \to \mathbb{R}$. The index by the distribution P highlights that the expected benefit and the expected cost of criterion g vary with P, and in particular, whether a criterion g satisfies the budget constraint depends on which distribution P is of interest. When the distribution P is unknown, the sample analog $\widehat{B}_n(g)$ provides some evidence whether g is feasible under the unknown distribution P. The evidence, however, is inconclusive due to sampling uncertainty because the sample analog $\widehat{B}_n(g)$ is an imperfect measure for the budget function $B(g; P)$.

2.2 Desirable properties for budget-constrained statistical rules

Denote by $\widehat{g}$ a statistical rule that selects an eligibility criterion after observing some experimental data of sample size n. This section first provides formal definitions for two desirable properties of $\widehat{g}$, and then proves it is impossible for any statistical rule to satisfy both properties when the data distribution is unknown and belongs to a sufficiently rich class of distributions $\mathcal{P}$. Importantly, an obvious extension to the existing approach in the EWM literature does not satisfy either property.

Definition 1. A statistical rule $\widehat{g}$ is *pointwise asymptotically optimal* under the data distribution P if for any $\epsilon > 0$

$$\lim_{n \to \infty} Pr_{P^n} \{W(\widehat{g}; P) - W(g_P^*; P) < -\epsilon\} = 0, \tag{6}$$

and *uniformly asymptotically optimal* over the class of distributions $\mathcal{P}$ if for any $\epsilon > 0$

$$\lim_{n \to \infty} \sup_{P \in \mathcal{P}} Pr_{P^n} \{W(\widehat{g}; P) - W(g_P^*; P) < -\epsilon\} = 0.$$

A statistical rule is *pointwise asymptotically feasible* under the data distribution P if $Pr_{P^n}\{B(\widehat{g};P) > k\} \to 0$, and *uniformly asymptotically feasible* over the class of distributions $\mathcal{P}$ if

$$\sup_{P\in\mathcal{P}} Pr_{P^n}\{B(\widehat{g};P) > k\} \to 0. \tag{7}$$

■

While both are desirable properties, Appendix A.1 proves a negative result that it is impossible for a statistical rule to satisfy both properties when the data distribution is unknown and belongs to a sufficiently rich class of distributions $\mathcal{P}$. Thus, policymakers might want to consider statistical rules that satisfy one of these two properties. However, Appendix A.2 proves that the direct extension to the existing approach in the EWM literature is neither asymptotically optimal nor asymptotically feasible. I refer to the direct extension as the sample-analog rule, which solves the sample version of the population constrained optimization problem (2):

$$\widehat{g}_s \in \arg\max_{\widehat{B}_n(g)\le k} \widehat{W}_n(g). \tag{8}$$

The subscript s emphasizes how this approach verifies whether a criterion satisfies the constraint by comparing the *sample* analog $\widehat{B}_n(g)$ with k directly, i.e. imposes a sample-analog constraint. Since the sample-analog rule $\widehat{g}_s$ restricts attention to criteria that satisfy the sample-analog constraint, there is no guarantee the selected criterion is actually feasible. This explains why the sample-analog rule is not asymptotically feasible. Similarly, there is no guarantee that the constrained optimal criterion satisfies the sample-analog constraint, and the sample-analog rule can select a suboptimal criterion. This explains why the sample-analog rule is not asymptotically optimal. Sections 2.3 and 2.4 propose statistical rules alternative to the sample-analog rule $\widehat{g}_s$, that are either uniformly asymptotically optimal or uniformly asymptotically feasible, respectively. The uniformity is considered for the class of data distributions $\mathcal{P}$ that satisfies the following assumption.

Assumption 1. *Estimation quality.* Uniformly over $P \in \mathcal{P}$, the recentered empirical processes $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ defined in (5) converge to mean-zero Gaussian processes G^W_P and G^B_P uniformly over $g \in \mathcal{G}$, with covariance functions $\Sigma^W_P(\cdot,\cdot)$ and $\Sigma^B_P(\cdot,\cdot)$ respectively. The covariance functions are uniformly bounded, with diagonal entries bounded away from zero uniformly over $g \in \mathcal{G}$. There is a uniformly consistent estimator $\widehat{\Sigma}^B(\cdot,\cdot)$ of the covariance function $\Sigma^B_P(\cdot,\cdot)$.

2.3 New statistical rule that ensures uniform asymptotic feasibility

This section proposes the mistake-controlling rule, a statistical rule that is uniformly asymptotically feasible. For a critical value α, it is defined as

$$\widehat{g}_{prob} \in \arg\max_{g\in\hat{\mathcal{G}}_\alpha} \widehat{W}_n(g). \tag{9}$$

Here the subscript $prob$ highlights that with high probability this statistical rule selects feasible criterion. The mistake-controlling rule restricts attention to a data-dependent class of eligibility criteria:

$$\hat{\mathcal{G}}_\alpha = \left\{ g : g \in \mathcal{G} \text{ and } \frac{\sqrt{n}\left(\widehat{B}_n(g) - k\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \le c_\alpha \right\}, \tag{10}$$

where c_α is the α-quantile from $\inf_{g\in\mathcal{G}} \left| \frac{G^B_P(g)}{\Sigma^B_P(g,g)^{1/2}} \right|$ for G^B_P the Gaussian process defined in Assumption 1, and $\widehat{\Sigma}^B(\cdot,\cdot)$ the consistent estimator for its covariance function. If the set $\hat{\mathcal{G}}_\alpha$ is empty, then I set $\widehat{g}_{prob}$ to not assign any eligibility: $\widehat{g}_{prob}(x) = 0$ for all $x \in \mathcal{X}$.

Note that the sample-analog constraint is tightened by $c_\alpha \cdot \frac{\widehat{\Sigma}^B(g,g)^{1/2}}{\sqrt{n}}$ where c_α is negative, which means the class $\hat{\mathcal{G}}_\alpha$ only includes eligibility criteria where the constraint is slack in the sample. The sample-analog constraint is tightened proportionally to the standard deviation to reflect the noise in $\widehat{B}_n(g)$. The degree of tightening scales inversely with the (square root of) sample size. Intuitively, larger sample size can reduce the sampling uncertainty. Unlike $\widehat{g}_s$, with probability at least $1-\alpha$ this statistical rule is guaranteed to not mistakenly choose infeasible eligibility criteria. The next theorem details the improvement by $\widehat{g}_{prob}$ over $\widehat{g}_s$ in terms of uniform asymptotic feasibility.

Theorem 1. Suppose Assumption 1 holds for the class of data distributions $\mathcal{P}$. If $\alpha_n \to 0$ as $n \to \infty$, then the mistake-controlling rule $\widehat{g}_{prob}$ defined in (9) is uniformly asymptotically feasible over $\mathcal{P}$.

2.4 New statistical rule that ensures uniform asymptotic optimality

Exceeding the budget constraint in the population might be desirable if it achieves higher welfare and does not cost too much. Note that the original constrained optimization problem (8) may be reformulated as

$$\max_{g\in\mathcal{G}} \inf_{\lambda\geq 0} W(g;P) - \lambda \cdot (B(g;P) - k). \tag{11}$$

where λ measures the marginal gain of relaxing constraint. This formulation implies that policymakers are willing to enforce the constraint at all costs since λ is unbounded. This formulation may not reflect the real objective, however, as policymakers might only be willing to trade off violations of constraint against gains in welfare to certain extent, bounding $\lambda \in [0, \overline{\lambda}]$. This upper bound results in a new objective function

$$\max_{g\in\mathcal{G}} \min_{\lambda\in[0,\overline{\lambda}]} W(g;P) - \lambda \cdot (B(g;P) - k). \tag{12}$$

Given this new objective function, I propose the trade-off statistical rule, defined as

$$\widehat{g}_r \in \arg\max_{g\in\mathcal{G}} \widehat{W}_n(g) - \overline{\lambda} \cdot (\widehat{B}_n(g) - k)_+, \tag{13}$$

where the subscript r highlights that this statistical rule is able to relax the constraint.

Theorem 2. Suppose Assumption 1 holds for the class of data distributions $\mathcal{P}$. Then the trade-off rule $\widehat{g}_r$ defined in (13) is uniformly asymptotically optimal under $\mathcal{P}$.

3 Experiments

3.1 Simulation

To ensure the practical relevance of the simulation, I attempt to preserve the distribution of the data from the OHIE. The benefit is defined to be the increase in the probability of reporting good subjective health after receiving Medicaid eligibility, and the cost is defined to be the per enrollee *excess* cost of Medicaid relative to the current expansion criterion. I defer the details on the construction of the estimates (Γ_i^*, R_i^*) to Appendix B. Table 1 presents the basic summary statistics for the estimates (Γ_i^*, R_i^*). For the purpose of this simulation study, the OHIE represents the population P, and therefore I can take these estimates as the true benefit and cost (Γ, R). Under the current definition of the cost, a criterion is feasible if it incurs a negative cost.

Table 1: Summary statistics of the OHIE sample by number of children

Number of children	Sample size	Sample mean of Γ_i^*	Sample mean of R_i^*
0	5,758	3.1%	\$651
1	1,736	10.3%	\$348
≥ 2	2,641	1.5%	-\$275
-	10,135	3.9%	\$358

Notes: This table presents summary statistics on the sample of individuals who responded to both the initial and the main surveys from the Oregon Health Insurance Experiment (the OHIE sample). The first three rows represent individuals living with different number of children (family members under age 19), and the last row is the aggregate.

I consider a criterion class $\mathcal{G}$ that includes income thresholds that can vary with number of children. The maximum feasible welfare is given by $W(g_P^*; P) = 3.8\%$, an increase of 3.8% in reporting good subjective health. The cost associated with the constrained optimal criterion is $B(g_P^*; P) = -\$0.6$, meaning per enrollee g_P^* costs \$0.6 less than the current expansion criterion.

Table 2 compares the performance of various statistical rules $\widehat{g}$ through 500 Monte Carlo iterations. At each iteration, I randomly draw observations from the OHIE sample to form a random sample. I simulate with the same sample size as the original sample to hold the amount of sampling uncertainty constant.

Table 2: Simulation results: asymptotic properties of statistical rules $\widehat{g}$

Statistical rule	sample-analog $\widehat{g}_s$	mistake-controlling $\widehat{g}_{prob}$	trade-off $\widehat{g}_r$
Prob. of selecting infeasible criteria	35.4%	8%	79.6%
Prob. of selecting suboptimal criteria	870%	98.6%	37.6%
Average welfare loss	0.06	0.60	-0.02
Average cost	-\$3	-\$57	\$105

Notes: This table reports asymptotic properties of statistical rules $\widehat{g}$, as averaged over 500 simulations. Row 1 reports the probability that the rule selects an eligibility criterion that violates the budget constraint, i.e. $Pr_{P^n}\{B(\widehat{g};P) > 0\}$. Row 2 reports the probability that the rule achieves strictly less welfare than the constrained optimal criterion g^*_P, i.e. $Pr_{P^n}\{W(\widehat{g};P) < W(g^*_P;P)\}$. Row 3 reports the average welfare loss of the rule relative to the maximum feasible welfare, i.e. $\frac{E_{P^n}[W(g^*_P;P)-W(\widehat{g};P)]}{W(g^*_P;P)}$. Row 4 reports the average cost of the criteria selected by the rule, i.e. $E_{P^n}\left[B(\widehat{g};P)\right]$.

Row 1 of Table 2 illustrates that it is possible for all three statistical rule $\widehat{g}$ to select infeasible criteria. A lower probability of selecting infeasible criteria suggests the rule is closer to achieving asymptotic feasibility. Proposition 1 describes distributions where the sample-analog rule $\widehat{g}_s$ is not asymptotic feasible. In the distribution calibrated to the OHIE sample, the sample-analog rule $\widehat{g}_s$ might not be asymptotically feasible, either, as it can select infeasible eligibility criteria in 19.4% of the draws. In contrast, Theorem 1 guarantees that the mistake-controlling rule $\widehat{g}_{prob}$ selects infeasible eligibility criteria in less than 5% of the draws, regardless of the distribution. Simulation confirms such guarantee as the mistakes only happen 8% of the time.

Row 2 of Table 2 illustrates that it is possible for all three statistical rule $\widehat{g}$ to achieve weakly higher welfare than the constrained optimal criterion g^*_P. This can happen when $\widehat{g}$ selects an infeasible criterion. A lower probability of selecting suboptimal criteria suggests the rule is closer to achieving asymptotic optimality. Theorem 2 implies that the trade-off rule $\widehat{g}_r$ is uniformly asymptotically optimal while there is no such guarantee for the sample-analog rule $\widehat{g}_s$.

In the distribution calibrated to the OHIE, the trade-off rule $\widehat{g}_r$ on average achieves higher welfare than the sample-analog rule $\widehat{g}_s$. As shown in row 3 of Table 2, the welfare loss of $\widehat{g}_r$ is -2% of the maximum feasible welfare $W(g^*_P;P)$, compared to 6% for $\widehat{g}_s$. However, its improvement can be at the cost of violating the budget constraint more often than $\widehat{g}_s$, at a rate of 79.6%. Comparing the distribution of the cost $B(\widehat{g}_s;P)$ and $B(\widehat{g}_r;P)$, I note whenever the trade-off rule $\widehat{g}_r$ selects an infeasible criterion, the amount of violation is small, so that on average the budget constraint would not be violated as shown in row 4 of Table 2. As a result, even though the trade-off rule $\widehat{g}_r$ is more likely to select infeasible eligibility criteria than the sample-analog rule $\widehat{g}_s$, the cost to these violations is limited.

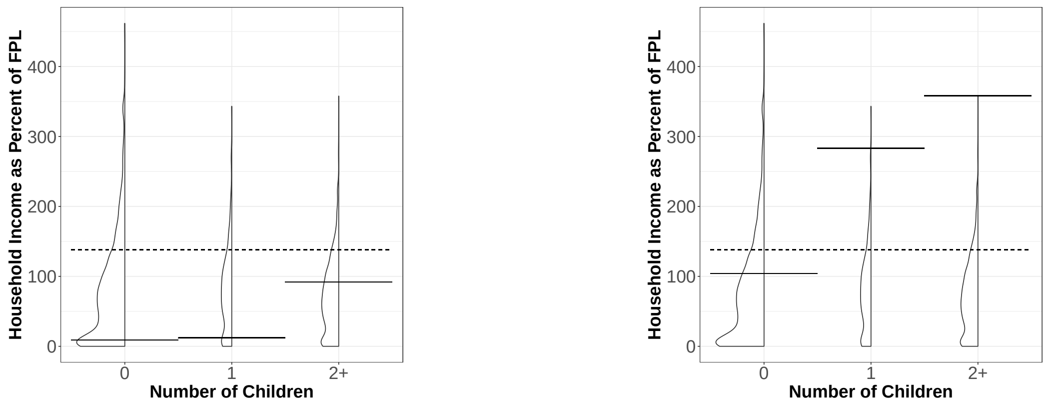
3.2 Application: budget-constrained Medicaid expansion

Figure 1 reports the Medicaid expansion criteria selected by the two statistical rules proposed in Sections 2.3 and 2.4 using results from the OHIE, allowing the income thresholds to vary with number of children. I defer the implementation details to Appendix B. The mistake-controlling rule $\widehat{g}_{prob}$ chooses to restrict Medicaid eligibility, especially lowering the income threshold for childless individuals far below the current level. This reflects the large variation in the cost estimates, which results in uncertainty about whether it would be feasible to provide eligibility to many individuals. In contrast, the trade-off rule $\widehat{g}_r$ chooses to assign Medicaid eligibility to more individuals, and to raise the income thresholds above the current level. The higher level occurs because on average the benefit estimates are positive, as illustrated in Table 1, which suggests many individuals still exhibit health benefit from being eligible for Medicaid. Under a reasonable assumption for an upper bound on the monetary value on the health benefit, the trade-off rule finds that the additional health benefit from violating the budget constraint exceeds the cost of doing so.

Figure 1: More flexible Medicaid expansion criteria

(a) criterion selected by the mistake-controlling rule

(b) criterion selected by the trade-off rule



Notes: The horizontal dashed line marks the income thresholds under the current expansion criterion, which is 138% regardless of the number of children in a household. The horizontal solid lines mark the more flexible criterion selected by various statistical rules, i.e. income thresholds that can vary with number of children. For each number of children, I also plot the underlying income distribution to visualize individuals below the thresholds.

4 Conclusion

In this paper, I extend the existing EWM approach to allow for a budget constraint where the cost of implementing any given eligibility criterion needs to be estimated. The existing EWM approach directly maximizes a sample analog of the social welfare function, and only accounts for constraints that can be verified with certainty in the larger population. In reality, the cost of providing eligibility to any given individual might be unknown ex-ante due to imperfect take-up and heterogeneity. Therefore, one cannot verify with certainty whether any given eligibility criterion satisfies the budget constraint in the population. I propose two alternative statistical rules: the mistake-controlling rule and the trade-off rule. The mistake-controlling rule is uniformly asymptotically feasible, namely with high probability it chooses eligibility criteria that satisfy the budget constraint in the population. The trade-off rule is uniformly asymptotically optimal, namely with high probability it chooses eligibility criteria that achieve at least the maximized feasible welfare. While the selected criterion may violate the budget constraint, the trade-off rule accounts for the per unit welfare gain from violating the constraint. I illustrate their asymptotic properties and implementation details using experimental data from the OHIE.

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A Impossibility results

A.1 No rule is both uniformly asymptotically optimal and uniformly asymptotically feasible

This section presents a negative result that applies when the data distribution is unknown and belongs to a sufficiently rich class of distributions $\mathcal{P}$: no statistical rule can be both uniformly asymptotically optimal and uniformly asymptotically feasible in this class. Assumption 2 explains the notion of richness: the sets of feasible eligibility criteria differ only marginally at nearby pairs of distributions. Assumptions 3 and 4 characterize distributions where such richness can be problematic for statistical rules to simultaneously achieve uniform asymptotic optimality and uniform asymptotic feasibility, as formalized in Theorem 3.

Assumption 2. *Contiguity.* There exists a distribution $P_0 \in \mathcal{P}$ under which a non-empty set of eligibility criteria satisfies the constraint exactly $\mathcal{G}_0 = \{g : B(g; P_0) = k\}$. Furthermore, the class of distributions $\mathcal{P}$ includes a sequence of data distributions $\{P_{h_n}\}$ contiguous to P_0, under which for all $g \in \mathcal{G}_0$, there exists some $C > 0$ such that

$$\sqrt{n} \cdot (B(g; P_{h_n}) - k) > C.$$

Assumption 3. *Binding constraint.* Under the data distribution P_0, the constraint is satisfied exactly at the constrained optimum i.e. $B(g^*_P; P_0) = k$.

Assumption 4. *Separation.* Under the data distribution P_0, $\exists \epsilon > 0$ such that for any feasible criterion g, whenever

$$\left|W(g; P_0) - W(g^*_{P_0}; P_0)\right| < \epsilon,$$

we have

$$\rho\left(g - g^*_{P_0}; P_0\right) = 0$$

for the semi-metric distance between two eligibility criteria:

$$\rho\left(g - g'; P\right) = Pr_P\left\{g(X) \neq g'(X)\right\}.$$

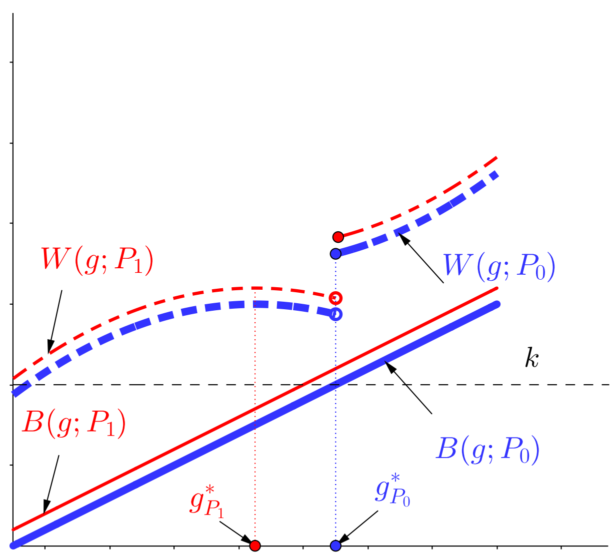
Theorem 3. Suppose Assumption 2 holds for the class of data distributions $\mathcal{P}$. For $P_0 \in \mathcal{P}$ considered in Assumption 2, suppose it also satisfies Assumptions 3 and 4. Then no statistical rule can be both uniformly asymptotically optimal and uniformly asymptotically feasible. In particular, if a statistical rule $\widehat{g}$ is pointwise asymptotically optimal and pointwise asymptotically feasible under P_0, then it is not uniformly asymptotically feasible.

Remark 1. Appendix C includes the proof of Theorem 3. In Appendix D.2, I give more primitive assumptions under which Assumption 2 is guaranteed to hold. These primitive assumptions are relatively weak. Furthermore, it is not implausible for real-world distributions to satisfy both Assumptions 3 and 4. This suggests the impossibility results are relevant in real-world settings.

Figure 2 provides some intuition for Theorem 3. The pictured distribution P_0 satisfies both Assumptions 3 and 4: the constrained optimal criterion $g^*_{P_0}$ satisfies the budget constraint exactly, i.e. $B(g^*_{P_0}; P_0) = k$, and is separated from the rest of feasible eligibility criteria. Note that if a statistical rule $\widehat{g}$ is pointwise asymptotically optimal and pointwise asymptotically feasible under P_0, then it has to select eligibility criteria close to $g^*_{P_0}$ with high probability over repeated sample draws distributed according to P_0^n as $n \to \infty$.

Under Assumption 2, the class of distributions $\mathcal{P}$ is sufficiently rich so that along a sequence of data distributions $\{P_{h_n}\}$ that is contiguous to P_0 as $n \to \infty$, the budget functions $B(g; P_{h_n})$ converge to $B(g; P_0)$ while $B(g^*_{P_0}; P_{h_n}) > k$, i.e. $g^*_{P_0}$ is not feasible under P_{h_n}. Figure 2 showcases P_1 as one distribution from this sequence. The contiguity between $\{P_{h_n}\}$ and P_0 implies that the statistical rule $\widehat{g}$ must select criteria close to $g^*_{P_0}$ with high probability under $P^n_{h_n}$ as well. However, the criterion $g^*_{P_0}$ is infeasible under P_{h_n}, and therefore the statistical rule $\widehat{g}$ cannot be asymptotically feasible under P_{h_n}. $\triangle$

Figure 2: Illustration for Assumptions 2-4 underlying Theorem 3



Notes: This figure plots welfare functions $W(g; P)$ (dashed lines) and budget functions $B(g; P)$ (solid lines) for populations distributed according to P_0 (blue thicker lines) or P_1 (red thin lines), where P_1 is a distribution from the sequence of distributions $\{P_{h_n}\}$ that is contiguous to P_0 under Assumption 2. The distribution P_0 satisfies Assumptions 3 and 4. The x-axis indexes a one-dimensional criterion class $\mathcal{G}$, e.g. eligibility criteria based on income thresholds, so that eligibility criteria can be ordered on $\mathbb{R}$. The black dotted line marks the budget threshold k. The bold blue dot marks $g^*_{P_0}$, the constrained optimal eligibility criterion under P_0. The bold red dot marks $g^*_{P_1}$, the constrained optimal eligibility criterion under P_1.

A.2 Sample-analog rule is neither asymptotically optimal nor asymptotically feasible

The previous negative result implies that no statistical rule can be both uniformly asymptotically optimal and uniformly asymptotically feasible. Thus, policymakers might want to consider statistical rules that satisfy one of these two properties. This section demonstrates why the direct extension to the existing approach in the EWM literature, the sample-analog rule (8), does not satisfy either property.

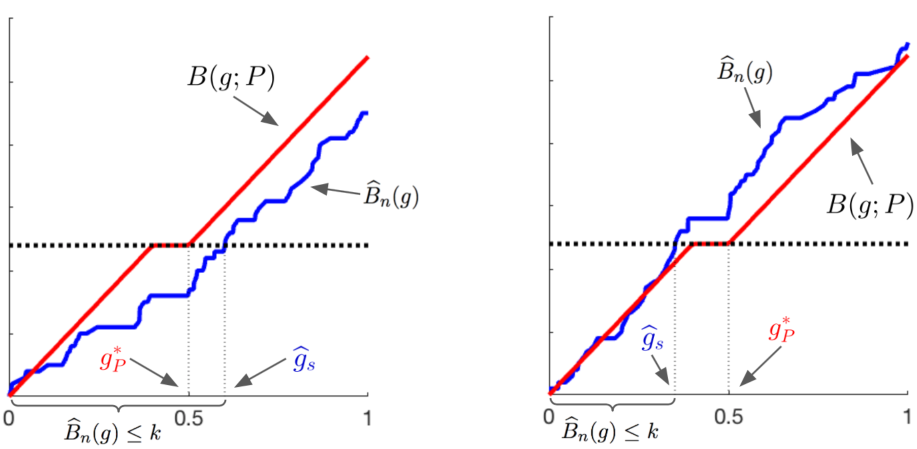
A key insight from Kitagawa and Tetenov (2018) is that without a constraint, the sample-analog rule is uniformly asymptotically optimal. Unfortunately this intuition does not extend to the current setting where the constraint involves an unknown cost. In Proposition 1 I find reasonable data distributions where the sample-analog rule is neither pointwise asymptotically optimal nor pointwise asymptotically feasible.

Proposition 1. Consider a one-dimensional criterion class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$, which is based on thresholds of a one-dimensional continuous characteristic X. Consider the special case where under distribution P, the benefit $\Gamma > 0$ almost surely and the cost R is binary with $R = 0$ for $X \in [\underline{t}, \overline{t}]$, but $Pr_P\{R = 1 \mid X\} \in (0, 1)$ otherwise. Then for some budget threshold $k = E_P[R \cdot \mathbf{1}\{X \leq \underline{t}\}]$, the budget constraint binds, and the sample-analog rule $\widehat{g}_s$ is neither asymptotically optimal nor asymptotically feasible under P.

Remark 2. The driving force behind Proposition 1 is that due to sampling uncertainty, whether a criterion satisfies the sample-analog constraint is an imperfect measure of whether it satisfies the constraint in the population. Figure 3 illustrates the setup of Proposition 1, where the sampling uncertainty can be particularly problematic for the sample-analog rule.

Figure 3: Illustration for Proposition 1

(a) infeasible (b) suboptimal



Notes: This figure plots the budget function $B(g; P)$ (red solid line) and its sample-analogs $\widehat{B}_n(g)$ (blue wriggly line) based on two different observed samples in panel (a) and (b) respectively. The x-axis indexes a one-dimensional criterion class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$ for a one-dimensional continuous characteristic X_i with support on $[0, 1]$. The black dotted line marks the budget threshold k. The constrained optimal threshold g_P^* is $t = 0.5$. The sample-analog rule $\widehat{g}_s$ selects an infeasible threshold in panel (a) and selects a suboptimal threshold in panel (b).

Since the sample-analog rule $\widehat{g}_s$ restricts attention to criteria that satisfy the sample-analog constraint, there is no guarantee the selected criterion is actually feasible. This is very likely to happen when there is welfare gain in exceeding the budget constraint as in the setup of Proposition 1 where $W(g; P)$ is strictly increasing in g. Therefore, the sample-analog rule $\widehat{g}_s$ is not asymptotically feasible under P. As illustrated in Figure 3, after observing a sample depicted in panel (a), the sample-analog rule picks an infeasible threshold because the sample-analog constraint is still satisfied there.

Similarly, there is uncertainty about whether the sample-analog constraint is satisfied at the constrained optimum g_P^*, and it is possible for the sample-analog rule $\widehat{g}_s$ to miss g_P^*. In the setup of Proposition 1, when the sample-analog rule $\widehat{g}_s$ misses g_P^*, it is guaranteed to select a suboptimal criterion and therefore is not asymptotically optimal under P. As illustrated in Figure 3, after observing a sample depicted in panel (b), the sample-analog rule picks a suboptimal threshold because the sample-analog constraint is violated at the constrained optimum g_P^*. $\triangle$

B Implementation details

B.1 Data

I use the experimental data from the OHIE, where Medicaid eligibility (D_i) was randomized in 2007 among Oregon residents who were low-income adults, but previously ineligible for Medicaid, and who expressed interest in participating in the experiment. Finkelstein et al. (2012) include a detailed description of the experiment and an assessment of the average effects of Medicaid on health and health care utilization. I include a cursory explanation here for completeness, emphasizing that my final sample differs from the original OHIE sample as I focus on individuals who responded both to the initial and the main surveys from the OHIE.

The original OHIE sample consists of 74,922 individuals (representing 66,385 households). Of these, 26,423 individuals responded to the initial mail survey, which collects information on income as percentage of the federal poverty level and number of children.[2] Recall that I want to consider a more flexible expansion criterion that allows the income threshold to vary with number of children. Therefore, the initial survey include the characteristics X of interest, and I exclude individuals who did not respond to the initial survey from my sample.

Due to this difference, the expansion criteria selected using my sample do not directly carry their properties to the population underlying the original OHIE sample, as the distributions of X differ.

The main survey collects data related to health (Y_i), health care utilization (C_i) and actual enrollment in Medicaid (M_i), which allows me to construct estimates for the other two inputs, namely (Γ, R). Therefore I further exclude individuals who did not respond to the main survey from my sample. More details on the main survey data and the construction of these estimates follow.

For health (Y_i), I follow the binary measurement in Finkelstein et al. (2012) based on self-reported health, where an answer of "poor/fair" is coded as $Y_i = 0$ and "excellent/very good/good" is coded as $Y_i = 1$. For health care utilization (C_i), the study collected measures of utilization of prescription drugs, outpatient visits, ER visits, and inpatient hospital visits. Finkelstein et al. (2012) annualize these utilization measures to turn these into spending estimates, weighting each type by its average cost (expenditures) among low-income publicly insured non-elderly adults in the Medical Expenditure Survey (MEPS). Note that health and health care utilization are not measured at the same scale, which requires rescaling when I consider the trade-off between the two. Lastly, since the enrollment in Medicaid still requires an application, not everyone eligible in the OHIE eventually enrolled in Medicaid, which implies $M_i \leq D_i$.

Given the setup of the OHIE, Medicaid eligibility (D_i) is random conditional on household size (number of adults in the household) entered on the lottery sign-up form and survey wave. While the original experimental setup would ensure randomization given household size, the OHIE had to adjust randomization for later waves of survey respondents (see the Appendix of Finkelstein et al. (2012) for more details). Denote the confounders (household size and survey wave) with V_i, and define the propensity score as $p(V_i) = Pr\{D_i = 1 \mid V_i\}$. If the propensity score is known, then the construction of the estimates follows directly from the formula (3). However, the adjustment for later survey waves means I need to estimate the propensity score, and I adapt the formula (3) following Athey and Wager (2020) to account for the estimated propensity score.

Specifically, define the conditional expectation function (CEF) of a random variable U_i as $\gamma^U = E[U_i \mid V_i, D_i]$. Since V_i in my case is discrete, I use a fully saturated model to estimate the propensity score $\widehat{p}(V_i)$ and the CEF $\widehat{\gamma}^U(V_i, D_i)$. I then form the estimated Horvitz-Thompson weight with the estimated propensity score as $\widehat{\alpha}(V_i, D_i) = \frac{D_i}{\widehat{p}(V_i)} - \frac{1-D_i}{1-\widehat{p}(V_i)}$. Define $\Gamma_i^* = \widehat{\gamma}^Y(V_i, 1) - \widehat{\gamma}^Y(V_i, 0) + \widehat{\alpha}(V_i, D_i) \cdot \big(Y_i - \widehat{\gamma}^Y(V_i, D_i)\big)$. Then as shown in the Appendix D.3, the estimate Γ_i^* is a noisy version of Γ, the health benefit due to Medicaid eligibility, and the noise is asymptotically unimportant.

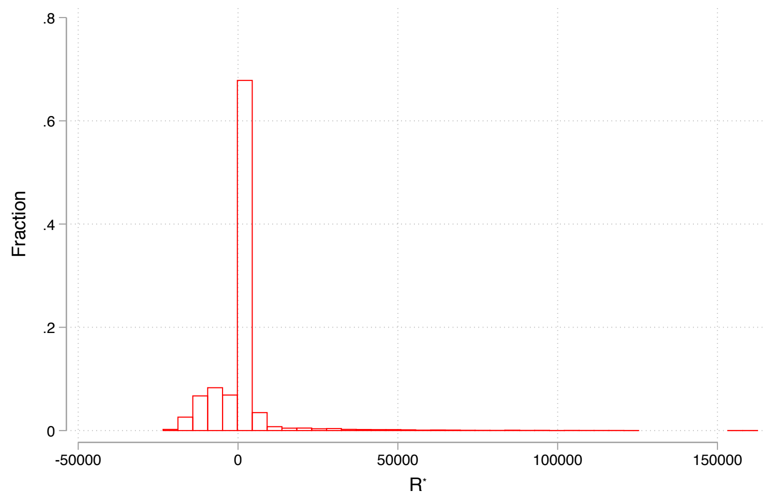
The 2014 Medicaid spending was roughly \$6,000 per adult enrollee in Oregon, according to the expenditure information obtained from MACPAC (2019). To formalize the budget constraint that the per enrollee cost of the proposed criterion cannot exceed the 2014 criterion, I need to account for imperfect take-up because not everyone eligible for Medicaid would enroll. Define $R_i^* =$

[2] More accurately, I follow Sacarny et al. (2020) to approximate number of children by the number of family members under age 19 living in house as reported on the initial mail survey.

$\widehat{\gamma}^Z(V_i, 1) + \frac{D_i}{\widehat{p}(W_i)} \cdot \left(Z_i - \widehat{\gamma}^Z(V_i, D_i)\right)$ where $Z_i = C_i - \$6,000 \cdot M_i$. Then as shown in the Appendix D.3, the estimate R_i^* is a noisy version of R, the per enrollee *excess* cost relative to the current level, and the noise is asymptotically unimportant.

Figure 4 plots the histogram of R_i^* to illustrate its heterogeneity. The histogram also visualizes the uncertainty about estimating the budget constraint. The more variation R_i^* has in the sample, relative to the sample size, the more estimation error there is in estimating the budget constraint.

Figure 4: Histogram of R_i^*



Notes: This figure plots the histogram of the estimated cost R_i^* in the OHIE sample, which is an estimate for an individual's cost to the Medicaid, in excess to the cost the per enrollee under the current expansion criterion.

B.2 Controlling the mistake with $\widehat{g}_{prob}$

To construct the mistake-controlling rule $\widehat{g}_{prob}$ as proposed in Section 2.3, I maximize the sample welfare function among eligibility criteria in $\hat{\mathcal{G}}_\alpha$, which are guaranteed to contain only feasible eligibility criteria with probability approaching $1 - \alpha$. For a conventional level of $\alpha = 5\%$, constructing $\hat{\mathcal{G}}_\alpha$ requires an estimate for the critical value c_α, the 5%-quantile from $\inf_{g \in \mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}}$, the infimum of the Gaussian process $G_P^B(\cdot) \sim \mathcal{GP}(0, \Sigma_P^B(\cdot, \cdot))$. In practice, I construct a grid on $\mathcal{G}$ as

$$\widetilde{\mathcal{G}} = \{g(x) = \mathbf{1}\{income \cdot 1\{numchild = j\} \leq y_j\} : j \in \{0, 1, \geq 2\}, y_j \in \{0, 50, 100, \dots, 500\}\} \tag{14}$$

for characteristics $x = (income,\ numchild)$. This grid thus consists of income thresholds every 50% of the federal poverty level, and the thresholds can vary with number of children. I then approximate the infimum over infinite-dimensional $\mathcal{G}$ by the minimum over $\widetilde{\mathcal{G}}$ with estimated covariance i.e. $\min_{g \in \widetilde{\mathcal{G}}} \frac{\widetilde{h}(g)}{\widehat{\Sigma}^B(g,g)^{1/2}}$. Here $\widetilde{h}(\cdot) \sim \mathcal{N}(0, \widehat{\Sigma}^B)$ is a Gaussian vector indexed by $g \in \widetilde{\mathcal{G}}$, with $\widehat{\Sigma}^B$ is submatrix of the covariance estimate $\widehat{\Sigma}^B(\cdot, \cdot)$ for $g \in \widetilde{\mathcal{G}}$. Based on 10,000 simulation draws I estimate c_α to be -2.56. The validity of this approximation is given by the uniform consistency of the covariance estimator under Assumption 1.

B.3 Controlling the trade-off with $\widehat{g}_r$

To construct the trade-off rule $\widehat{g}_{prob}$ as proposed in Section 2.4, I need to choose $\overline{\lambda}$, the upper bound on the marginal gain from violating the constraint. In my empirical illustration, the budget constraint is in terms of monetary value. The objective function, however, is measured based on self-reported health, which does not directly translate to a monetary value. Following Finkelstein et al. (2019), I convert self-reported health into value of a statistical life year (VSLY) based on existing estimates. Specifically, a conservative measure for the increase in quality-adjusted life year (QALY) when self-reported health increases from "poor/fair" to "excellent/very good/good" is roughly 0.6. The "consensus" estimate for the VSLY for one unit of QALY from Cutler (2004) is $100,000 for the general US population. Taken these estimates together, I set $\overline{\lambda} = 1/(0.6 \cdot 10^5)$.

C Proofs of theorems

Proof. **Proof of Theorem 3.** The larger population is a probability space $(\Omega, \mathcal{A}, P)$, which induces the sampling distribution P^n that governs the observed sample. A statistical rule $\widehat{g}$ is a mapping $\widehat{g}(\cdot) : \Omega \to \mathcal{G}$ that selects a policy from the policy class $\mathcal{G}$ based on the observed sample. Note that the selected policy $\widehat{g}(\omega)$ is still deterministic because the policy class $\mathcal{G}$ is restricted to be deterministic policies, though the proof below extends to random policies. When no confusion arises, I drop the reference to event ω for notational simplicity.

Suppose $\widehat{g}$ is asymptotically optimal and asymptotically feasible under P_0. I want to prove there is non-vanishing chance that $\widehat{g}$ selects policies that are infeasible some other distributions:

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n} \{\omega : B(\widehat{g}(\omega); P) > k\} > 0. \tag{15}$$

Asymptotical optimality under P_0 implies for any $\epsilon > 0$ we have

$$\limsup_{n\to\infty} Pr_{P_0^n} \left\{\omega : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega); P_0) > \epsilon\right\} = 0. \tag{16}$$

Asymptotical feasibility under P_0 implies $Pr_{P_0^n}\{\omega : B(\widehat{g}(\omega); P_0) \le k\} \to 1$. I first prove the event that $\widehat{g}$ selects a policy that meets the constraint exactly under distribution P_0 happens with probability approaching one under P_0 as $n \to \infty$: $Pr_{P_0^n} \{\omega : B(\widehat{g}(\omega); P_0) = k\} \to 1$. To see this, note by Law of Total Probability, we have the following lower bound

$$Pr_{P_0^n} \{\omega : B(\widehat{g}(\omega); P_0) = k\} \tag{17}$$
$$\geq Pr_{P_0^n} \left\{\omega : \rho\left(\widehat{g}(\omega) - g^*_{P_0}; P_0\right) = 0\right\} \cdot Pr_{P_0^n} \left\{B(\widehat{g}; P_0) = k \mid \rho\left(\widehat{g} - g^*_{P_0}; P_0\right) = 0\right\} \tag{18}$$

I next show this lower bound converges to one. For ϵ as specified in Equation (16), if we take it small enough to satisfy Assumption 4, then under the event ω where $|W(\widehat{g}(\omega); P) - W(g^*_P; P)| < \epsilon$ and $\widehat{g}(\omega)$ is feasible, we have $\rho\left(\widehat{g}(\omega) - g^*_{P_0}; P_0\right) = 0$. Specifically, the probability of such event has a lower bound

$$\begin{aligned} &Pr_{P_0^n} \left\{\omega : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega); P_0) \le \epsilon \; and \; B(\widehat{g}(\omega); P_0) \le k\right\} \\ \geq &Pr_{P_0^n} \left\{\omega : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega); P_0) \le \epsilon\right\} + Pr_{P_0^n} \{\omega : B(\widehat{g}(\omega); P_0) \le k\} - 1. \end{aligned}$$

The first two terms converge to one as $n \to \infty$ respectively under asymptotical optimality and feasibility, which implies $Pr_{P_0^n} \left\{\rho\left(\widehat{g} - g^*_{P_0}; P_0\right) = 0\right\} \to 1$.

Furthermore, by definition of $\rho(\cdot, \cdot; P_0)$ stated in Assumption 4, we know conditional on the event $\rho\left(\widehat{g}(\omega) - g^*_{P_0}; P_0\right) = 0$, we have $\widehat{g}(\omega)$ is equal to $g^*_{P_0}$ almost surely. Since the constraint is exactly satisfied at $g^*_{P_0}$, we must also have:

$$\begin{aligned} &Pr_{P_0^n} \left\{B(\widehat{g}; P_0) = k \mid \rho\left(\widehat{g} - g^*_{P_0}; P_0\right) = 0\right\} \\ = &Pr_{P_0^n} \left\{B(g^*_{P_0}; P_0) = k \mid \rho\left(\widehat{g} - g^*_{P_0}; P_0\right) = 0\right\} = 1. \end{aligned}$$

Following the notation in Assumption 2, denote the set of policies where the constraints bind exactly under the limit distribution P_0 by

$$\mathcal{G}_0 = \{g \in \mathcal{G} : B(g; P_0) = k\}. \tag{19}$$

Then the above argument shows $Pr_{P_0^n} \{A_n : B(\widehat{g}(A_n); P_0) = k\} \to 1$.

Under Assumption 2, the sequence $P^n_{h_n}$ is contiguous with respect to the sequence P^n_0 , which means $P^n_0(A_n) \to 0$ implies $P^n_{h_n}(A_n) \to 0$ for every sequence of measurable sets A_n on $\mathcal{A}^n$. also converges to one under $P^n_{h_n}$. Equivalently there exists an $N(u)$ such that for all $n \ge N(u)$, we have $Pr_{P^n_{h_n}} \{A_n : B(\widehat{g}(A_n); P_0) = k\} \ge 1-u$. That is, with high probability, the statistical rule $\widehat{g}$ selects policies from $\mathcal{G}_0$ based on the observed sample distributed according to $P_{h_n}^n$. Recall Assumption 2 implies for all $g \in \mathcal{G}_0$, for any sample size n, we have $B(g; P_{h_n}) - k > C/\sqrt{n}$. Thus this statistical rule cannot uniformly satisfy the constraint since with sample size $n \geq N(u)$, we have

$$\sup_{P \in \mathcal{P}} Pr_{P^n} \{B(\widehat{g}; P) > k\} \geq Pr_{P_{h_n}^n} \{B(\widehat{g}; P_{h_n}) > k\} \geq 1 - u \tag{20}$$

$\square$

Proof. **Proof of Proposition 1.** Consider a one-dimensional policy class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$, which includes thresholds for a one-dimensional continuous characteristic X. Since $\Gamma > 0$ almost surely, the welfare function $W(t; P) \coloneqq E_P[\Gamma \cdot \mathbf{1}\{X \leq t\}]$ is strictly increasing in t.

Suppose the budget constraint takes the form of a capacity constraint, involving a binary take-up decision R (that may or may not be independent of X). By assumption, the probability an individual takes up the treatment for $X \in [\underline{t}, \overline{t}]$ is zero but between zero and one otherwise. Then the budget function $B(t; P) \coloneqq E_P[R \cdot \mathbf{1}\{X \leq t\}]$ is flat in the interval $[\underline{t}, \overline{t}]$ but also strictly increasing otherwise.

The population problem is

$$\max_t \; W(t; P) \; s.t. \; B(t; P) \leq k,$$

where $B(t; P) = k$ for $t \in [\underline{t}, \overline{t}]$ by assumption. The constrained optimal threshold is therefore the highest threshold where the constraint is satisfied exactly i.e. $t^* = \overline{t}$. This also implies $R \cdot \mathbf{1}\{X \leq t\} \sim Bernoulli(k)$ for $t \in [\underline{t}, \overline{t}]$.

The sample-analog rule solves the following sample problem

$$\max_t \frac{1}{n} \sum_i \Gamma_i \cdot \mathbf{1}\{X_i \leq t\} \; s.t. \; \widehat{B}_n(t) \leq k$$

for $\widehat{B}_n(t) \coloneqq \frac{1}{n} \sum_i R_i \cdot \mathbf{1}\{X_i \leq t\}$. Given that $\Gamma > 0$ almost surely, the sample-analog rule equivalently solves $\max_t \widehat{B}_n(t) \; s.t. \; \widehat{B}_n(t) \leq k$. However, the solution is not unique because $\widehat{B}_n(t)$ is a step function. To be conservative, let the sample-analog rule be the smallest possible threshold to maximize $\widehat{B}_n(t)$:

$$\widehat{t} = \min \left\{ \arg\max_t \{ \widehat{B}_n(t) \; s.t. \; \widehat{B}_n(t) \leq k \} \right\}.$$

Note that we can also write $\widehat{B}_n(t) = \frac{1}{n} \sum_{R_i = 1} \mathbf{1}\{X_i \leq t\}$, which makes it clear that $\widehat{t}$ corresponds to ranking X_i among individuals with $R_i = 1$, and then picking the lowest threshold such that we assign treatment to the first $\lfloor k \cdot n \rfloor$ individuals. This also means if in the sample few individuals take up the treatment such that $\frac{1}{n} \sum_i R_i \leq k$, we can have a sample-analog rule that treats everyone up to $\max_{R_i = 1} X_i$. Taken together both scenarios, we note the sample-analog rule implies the treated share in the sample is equal to

$$\widehat{B}_n(\widehat{t}) \coloneqq \frac{1}{n} \sum_i R_i \cdot \mathbf{1}\{X_i \leq \widehat{t}\} = \min \left\{ \frac{1}{n} \sum_i R_i, \frac{\lfloor k \cdot n \rfloor}{n} \right\}.$$

Note that

$$\begin{cases} \widehat{t} > \overline{t} \Leftrightarrow & B(\widehat{t}; P) > B(\overline{t}; P) \\ \widehat{t} < \underline{t} \Leftrightarrow & W(\widehat{t}; P) < W(\underline{t}; P) \end{cases}$$

which means whenever $\widehat{t} > \overline{t}$, the sample-analog rule violates the constraint in the population as $B(\overline{t}; P) = k$; whenever $\widehat{t} < \underline{t}$, the sample-analog rule achieves strictly less welfare than t^* in the population because $W(\underline{t}; P)$ is strictly less than $W(t^*; P)$. We next derive the limit probability for these two events. Applying Law of Total Probability, we have

$$\begin{aligned}
& Pr_{P^n} \left\{ \widehat{B}_n(\overline{t}) < \widehat{B}_n(\widehat{t}) \right\} \\
=& Pr_{P^n} \left\{ \widehat{B}_n(\overline{t}) < \frac{\lfloor k \cdot n \rfloor}{n} \; and \; \frac{\lfloor k \cdot n \rfloor}{n} < \frac{1}{n} \sum_i R_i \right\} + Pr_{P^n} \left\{ \widehat{B}_n(\overline{t}) < \frac{1}{n} \sum_i R_i \; and \; \frac{\lfloor k \cdot n \rfloor}{n} \geq \frac{1}{n} \sum_i R_i \right\} \\
\geq& Pr_{P^n} \left\{ \widehat{B}_n(\overline{t}) < \frac{\lfloor k \cdot n \rfloor}{n} \; and \; \frac{\lfloor k \cdot n \rfloor}{n} < \frac{1}{n} \sum_i R_i \right\} \\
\geq& Pr_{P^n} \left\{ \widehat{B}_n(\overline{t}) < \frac{\lfloor k \cdot n \rfloor}{n} \right\} + Pr_{P^n} \left\{ \frac{\lfloor k \cdot n \rfloor}{n} < \frac{1}{n} \sum_i R_i \right\} - 1
\end{aligned} \tag{21}$$

For the first term in (21), we have the following lower bound

$$
\begin{aligned}
&Pr_{P^n}\left\{\frac{1}{n}\sum_i R_i \cdot \mathbf{1}\{X_i \leq \bar{t}\} \leq \frac{k \cdot n - 1}{n}\right\} \\
=&Pr_{P^n}\left\{\sqrt{n}\left(\frac{1}{n}\sum_i R_i \cdot \mathbf{1}\{X_i \leq \bar{t}\} - k\right) \leq -\frac{1}{\sqrt{n}}\right\} \to 0.5
\end{aligned}
$$

To see the convergence, we apply the Central Limit Theorem to the LHS, and note that $-\frac{1}{\sqrt{n}}$ converges to zero. Denote $p_R = Pr\{R = 1\}$. For the second term in (21), we have the following lower bound

$$
\begin{aligned}
&Pr_{P^n}\left\{\frac{1}{n}\sum_i R_i \geq \frac{k \cdot n}{n}\right\} \\
=&Pr_{P^n}\left\{\sqrt{n}\left(\frac{1}{n}\sum_i R_i - p_R\right) \geq \sqrt{n} \cdot (k - p_R)\right\} \to 1
\end{aligned}
$$

To see the convergence, we apply the Central Limit Theorem to the LHS, and note that $\sqrt{n} \cdot (k - p_R)$ diverges to $-\infty$ for $p_R > k$. We thus conclude

$$
\lim_{n\to\infty} Pr_{P^n}\left\{B(\widehat{t}; P) > B(\bar{t}; P)\right\} \geq 0.5
$$

which proves $\widehat{t}$ is not pointwise asymptotically feasible under the distribution P.

Similar argument shows $Pr_{P^n}\left\{\widehat{B}_n(t^*) > \widehat{B}_n(\underline{t})\right\}$ has a limit of one half. We thus conclude

$$
\lim_{n\to\infty} Pr_{P^n}\left\{W(\widehat{t}; P) < W(\underline{t}; P)\right\} \geq 0.5
$$

which proves that $\widehat{t}$ is not pointwise asymptotically optimal under the distribution P. $\square$

Proof. **Proof of Theorem 1.** By construction, the limit probability for any policy in $\hat{\mathcal{G}}_\alpha$ to violate the budget constraint is

$$
\begin{aligned}
&Pr_{P^n}\{\exists g : g \in \hat{\mathcal{G}}_\alpha \; and \; B(g;P) > k\} = Pr_{P^n}\{\min_{B(g;P)>k} \frac{\sqrt{n}\left(\widehat{B}_n(g) - k\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
\leq & Pr_{P^n}\{\min_{B(g;P)>k} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
\leq & Pr_{P^n}\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\}
\end{aligned}
$$

Under Assumption 1, uniformly over $P \in \mathcal{P}$, the empirical process $\left\{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)\right\}$ converges to a Gaussian process G_P^B for $G_P^B(\cdot) \sim \mathcal{GP}(0, \Sigma_P^B(\cdot,\cdot))$ and we have a consistent covariance estimate $\widehat{\Sigma}^B(\cdot,\cdot)$. Then by the definition of c_α, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n}\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
=& \sup_{P\in\mathcal{P}} Pr_{P^n}\{\inf_{g\in\mathcal{G}} \frac{G_P^B(g)}{\Sigma^B(g,g;P)^{1/2}} \leq c_\alpha\} = \alpha.
\end{aligned}
$$

Theorem 1 follows from the above argument, replacing α with α_n in its proof. Specifically, by construction we have

$$
\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n}\{B(\hat{g}_{prob};P) > k\} \leq \limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n}\{\exists g : g \in \hat{\mathcal{G}}_{\alpha_n} \; and \; B(g;P) > k\} \leq \alpha_n.
$$

Decompose the welfare loss into

$$
\begin{aligned}
& W(g_P^*;P) - W(\hat{g}_{prob};P) \\
=& W(g_P^*;P) - \widehat{W}_n(\hat{g}_{prob}) + \widehat{W}_n(\hat{g}_{prob}) - W(\hat{g}_{prob};P) \\
\leq & \sup_{g\in\mathcal{G}} |W(g;P) - \widehat{W}_n(g)| + \widehat{W}_n(g_P^*) - \widehat{W}_n(\hat{g}_{prob})
\end{aligned} \tag{22}
$$

Under the event that $g_P^* \in \hat{\mathcal{G}}_{\alpha_n}$, we are guaranteed that the last term is non-positive. This happens with probability

$$
Pr_{P^n}\{\frac{\sqrt{n}\left(\widehat{B}_n(g_P^*) - k\right)}{\widehat{\Sigma}^B(g_P^*, g_P^*)^{1/2}} \leq c_{\alpha_n}\} \overset{a}{\sim} \Phi\left(c_{\alpha_n} + \frac{\sqrt{n}\left(k - B(g_P^*;P)\right)}{\Sigma^B(g_P^*, g_P^*;P)^{1/2}}\right) \to 1 \tag{23}
$$

for $B(g_P^*;P)$ strictly below the threshold. Since c_{α_n} diverges to $-\infty$ at a rate slower $\sqrt{n}$, the term in the parenthesis diverges to ∞ as $n \to \infty$. I then apply Lemma 4 and 6 (proved in Section D.3 under primitive conditions that lead to Assumption 1), which shows $\sup_{g\in\mathcal{G}} |W(g;P) - \widehat{W}_n(g)|$ converges to zero in probability. I then conclude $\hat{g}_{prob}$ is asymptotically optimal under distribution P. □

Proof. **Proof of Theorem 2.** I first prove $\widehat{g}_r$ is asymptotically optimal. Denote the new objective function with $V(g;P) = W(g;P) - \overline{\lambda} \cdot (B(g;P) - k)_+$, whose maximizer is $\widetilde{g}_P$. Denote $\widehat{V}_n(g) = -\widehat{W}_n(g) + \overline{\lambda} \cdot (\widehat{B}_n(g) - k)_+$ to be the sample-analog, whose maximizer is $\widehat{g}_r$. Apply uniform deviation bound to the difference between the value under $\widehat{g}_r$ and $\widetilde{g}_P$, we have

$$
\begin{aligned}
V(\widehat{g}_r;P) - V(\widetilde{g}_P;P) &= V(\widehat{g}_r;P) - \widehat{V}_n(\widetilde{g}_P) + \widehat{V}_n(\widetilde{g}_P) - V(\widetilde{g}_P;P) && (24)\\
&\le V(\widehat{g}_r;P) - \widehat{V}_n(\widehat{g}_r) + \widehat{V}_n(\widetilde{g}_P) - V(\widetilde{g}_P;P) && (25)\\
&\le 2\sup_g \left| V(g;P) - \widehat{V}_n(g) \right| && (26)\\
&= 2\sup_g \left| W(g;P) - \widehat{W}_n(g) - \overline{\lambda} \cdot (\max\{B(g;P) - k, 0\} - \max\{\widehat{B}_n(g) - k, 0\}) \right| && (27)\\
&\le 2\sup_g \left| \widehat{W}_n(g) - W(g;P) \right| + 2\overline{\lambda} \cdot \sup_g \left| \max\{\widehat{B}_n(g) - k, 0\} - \max\{B(g;P) - k, 0\} \right| && (28)\\
&\le 2\sup_g \left| \widehat{W}_n(g) - W(g;P) \right| + 2\overline{\lambda} \cdot \sup_g \left| \widehat{B}_n(g) - B(g;P) \right| && (29)
\end{aligned}
$$

The last line uses the fact that $|\max\{a,0\} - \max\{b,0\}| \le |a-b|$. Both terms, $\sup_g \left| \widehat{W}_n(g) - W(g;P) \right|$ and $\sup_g \left| \widehat{B}_n(g) - B(g;P) \right|$ converge to zero in probability under Assumption 1. Specifically, Lemma 4 and 6 (proved in Section D.3 under primitive conditions that lead to Assumption 1) imply the uniform convergence in probability

$$
\sup_{P\in\mathcal{P}} \sup_{g\in\mathcal{G}} \left| \widehat{W}_n(g) - W(g;P) \right| \to_p 0, \ \sup_{P\in\mathcal{P}} \sup_{g\in\mathcal{G}} \left| \widehat{B}_n(g) - B(g;P) \right|_p \to 0.
$$

I thus conclude

$$
\sup_{P\in\mathcal{P}} |V(\widehat{g}_r;P) - V(\widetilde{g}_P;P)| \to_p 0.
$$

Furthermore, we have $W(g_P^*;P) \le V(\widetilde{g}_P;P) \le W(\widetilde{g}_P;P)$. By definition

$$
\begin{aligned}
W(g_P^*;P) = \max_{B(g;P)\le k} W(g;P) &= \max_{B(g;P)\le k} W(g;P) - \overline{\lambda} \cdot (B(g;P) - k)_+ && (30)\\
&\le \max_{g\in\mathcal{G}} W(g;P) - \overline{\lambda} \cdot (B(g;P) - k)_+ && (31)\\
&= W(\widetilde{g}_P;P) - \overline{\lambda} \cdot (B(\widetilde{g}_P;P) - k)_+ \le W(\widetilde{g}_P;P) && (32)
\end{aligned}
$$

Putting these together, we have

$$
\begin{aligned}
&\inf_{P\in\mathcal{P}} \{W(\widehat{g}_r;P) - W(g_P^*;P)\}\\
\ge &\inf_{P\in\mathcal{P}} \{V(\widehat{g}_r;P) - W(g_P^*;P)\}\\
\ge &\inf_{P\in\mathcal{P}} \{V(\widehat{g}_r;P) - V(\widetilde{g}_P;P)\} + \inf_{P\in\mathcal{P}} \{V(\widetilde{g}_P;P) - W(g_P^*;P)\}\\
\ge &\inf_{P\in\mathcal{P}} \{V(\widehat{g}_r;P) - V(\widetilde{g}_P;P)\} \to_p 0.
\end{aligned}
$$

which proves uniform asymptotic optimality. ∎

D Primitive assumptions and auxiliary lemmas

I first prove the optimal rule that solves the population constrained optimization problem takes the form of threshold. In Section D.2, I first provide primitive assumptions on the class of DGPs. I then prove in Lemma 1, which establishes that these primitive assumptions imply Assumption 2.

In Section D.3, I verify Assumption 1 for settings where the observed sample comes from an RCT or an observational study, and the propensity score can be estimated efficiently based on parametric regressions.

D.1 Optimal rule without functional form restriction

The population problem is to find rules based on X_i that solves

$$\max_{g:\mathcal{X}\to\{0,1\}} E[\Gamma_i g(X_i)] \; s.t. \; E[R_i g(X_i)] < k$$

By Law of Iterated Expectation, we can write the constrained optimization problem as

$$\max_{g:\mathcal{X}\to\{0,1\}} E[\gamma(X_i) g(X_i)] \; s.t. \; E[r(X_i) g(X_i)] < k$$

where $\gamma(X_i) = E[\Gamma_i \mid X_i]$ and $r(X_i) = E[R_i \mid X_i]$.

Claim 1. Let $d\mu = r(x)f(x)dx$ denote the positive measure. The constrained optimization problem is equivalent to

$$\max_{g:\mathcal{X}\to\{0,1\}} \int \frac{\gamma(x)}{r(x)} g(x) d\mu \; s.t. \; \int g(x) d\mu = k$$

Let X^* be the support of the solution g^*. It will take the form of $X^* = \{x : \frac{\gamma(x)}{r(x)} > c\}$ where c is chosen so that $\mu(X^*) = k$.

Proof. Let X be the support of any $g \neq g^*$ with $\mu(X) = k$. Then the objective function associated g is

$$\begin{aligned}
\int_{X^*} \frac{\gamma}{r} d\mu - \int_X \frac{\gamma}{r} d\mu &= \int \frac{\gamma(x)}{r(x)} \mathbf{1}\{x \in X^*\} d\mu - \int \frac{\gamma(x)}{r(x)} \mathbf{1}\{x \in X\} d\mu \\
&= \int \frac{\gamma(x)}{r(x)} \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu
\end{aligned}$$

By definition of X^*, we have $\frac{\gamma(x)}{r(x)} > c$ for $x \in X^* \backslash X$ and $\frac{\gamma(x)}{r(x)} < c$ for $x \in X \backslash X^*$. Also note that μ is a positive measure. Then the above difference is lower bounded by

$$\int \frac{\gamma(x)}{r(x)} \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu \geq c \int \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu \geq 0$$

as by construction, we have $\int \mathbf{1}\{x \in X^* \backslash X\} d\mu = 1$ since $\mu(X^*) = \mu(X) = k$. $\square$

D.2 Primitive assumptions for contiguity

Assumption 5. Assume the class of DGPs $\{P_\theta : \theta \in \Theta\}$ has densities p_θ with respect to some measure μ. Assume P_θ is DQM at P_0 i.e. $\exists \dot{\ell}_0$ s.t. $\int [\sqrt{p_h} - \sqrt{p_0} - \frac{1}{2} h' \dot{\ell}_0 \sqrt{p_0}]^2 d\mu = o(\left\|h^2\right\|)$ for $h \to 0$.

Assumption 6. For all policies g, $B(g; P_\theta)$ is twice continuously differentiable in θ at 0, and the derivatives are bounded from above and away from zero within an open neighborhood $\mathcal{N}_\theta$ of zero uniformly over $g \in \mathcal{G}$.

Lemma 1. Under Assumption 5, the class $\mathcal{P}$ includes a sequence of data distribution $\{P_{h_n}\}$ that is contiguous to P_0 for every h_n satisfying $\sqrt{n} h_n \to h$ e.g. take $h_n = h/\sqrt{n}$. This proves the first part of Assumption 2. Suppose further Assumption 6 holds, then there exists some h for the second part of Assumption 2 to hold,.

Proof. **Proof of Lemma 1.** By Theorem 7.2 of Vaart (1998), the log likelihood ratio process converges under P_0 (denoted with $\overset{p_0}{\rightsquigarrow}$) to a normal experiment

$$\log \prod_{i=1}^{n} \frac{p_{h_n}}{p_0}(A_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h'\dot{\ell}_0(A_i) - \frac{1}{2} h' I_0 h + o_{P_0}(1) \tag{33}$$

$$\overset{p_0}{\rightsquigarrow} \mathcal{N}\left(-\frac{1}{2}h' I_0 h, h' I_0 h\right) \tag{34}$$

where $\dot{\ell}_0$ is the score and $I_{P_0} = E_{P_0}[\dot{\ell}_0(A_i)\dot{\ell}_0(A_i)']$ exists. The convergence in distribution of the log likelihood ratio to a normal with mean equal to $-\frac{1}{2}$ of its variance in (34) implies mutual contiguity $P_0^n \lhd\rhd P_{h_n}^n$ by Le Cam's first lemma (see Example 6.5 of Vaart (1998)). This proves the first part of the lemma.

By Taylor's theorem with remainder we have for each policy g

$$B(g; P_{h/\sqrt{n}}) - B(g; P_0) = \frac{h'}{\sqrt{n}} \frac{\partial B(g; P_0)}{\partial \theta} + \frac{1}{2}\frac{1}{n} h' \frac{\partial^2 B(g; P_{\tilde{\theta}_n})}{\partial \theta \partial \theta'} h$$

where $\tilde{\theta}_n$ is a sequence of values with $\tilde{\theta}_n \in [0, h/\sqrt{n}]$ that can depend on g. Take h so that the first term is positive for policies with $B(g; P_0) = k$. Such h exists because we assume $\frac{\partial B(g;P_0)}{\partial \theta}$ is bounded away from zero. For $g \in \mathcal{G}_0$ where $\mathcal{G}_0 = \{g : B(g; P_0) = k\}$, the constraints are violated under P_{h_n} and furthermore (multiplying by $\sqrt{n}$)

$$\sqrt{n} \cdot (B(g; P_{h_n}) - k) > h' \frac{\partial B(g; P_0)}{\partial \theta} > 0$$

for every n. This proves the second part of the lemma for $C = \inf_{g \in \mathcal{G}_0} \left| h' \frac{\partial B(g;P_0)}{\partial \theta} \right|$. ☐

D.3 Primitive assumptions and proofs for estimation quality

In this section, I verify that $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$ satisfy Assumption 1 under primitive assumptions on the policy class and the OHIE.

Assumption 7. VC-class: The policy class $\mathcal{G}$ has a finite VC-dimension $v < \infty$.

To introduce the assumptions on the OHIE, I first recall the definitions of components in $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$:

$$\widehat{W}_n(g) \coloneqq \frac{1}{n} \sum_i \Gamma_i^* \cdot g(X_i), \; \widehat{B}_n(g) \coloneqq \frac{1}{n} \sum_i R_i^* \cdot g(X_i)$$

for the doubly-robust scores

$$\begin{aligned}
\Gamma_i^* &= \widehat{\gamma}^Y(V_i, 1) - \widehat{\gamma}^Y(V_i, 0) + \widehat{\alpha}(V_i, D_i) \cdot \left(Y_i - \widehat{\gamma}^Y(V_i, D_i)\right) \\
R_i^* &= \widehat{\gamma}^Z(V_i, 1) + \frac{D_i}{\widehat{p}(V_i)} \cdot \left(Z_i - \widehat{\gamma}^Z(V_i, D_i)\right)
\end{aligned}$$

and observed characteristics X_i. Here V_i collects the confounders in OHIE, namely household size and survey wave. Note that while X_i can overlap with V_i, the policy $g(X_i)$ needs not vary by V_i. In the OHIE example, the policy is based on number of children and income. However, conditional on household size and survey wave, number of children and income are independent of the lottery outcome in OHIE.

Recall that $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$ are supposed to approximate net benefit Γ and net excess cost R of Medicaid eligibility. I provide more precise definitions for Γ and R as the primitive assumptions are stated in terms of their components.

Let $Y(1)$ be the (potential) subjective health when one is given Medicaid eligibility, and $Y(0)$ be the (potential) subjective health when one is not given Medicaid eligibility. Recall the definition for

$$\Gamma = Y(1) - Y(0)$$

We only observe actual subjective health Y_i.

Let $M(1)$ be the (potential) enrollment in Medicaid when one is given Medicaid eligibility, and $M(0)$ be the (potential) enrollment in Medicaid when one is not given Medicaid eligibility.

Let $C(1)$ be the (potential) cost to the government when one is given Medicaid eligibility, and $C(0)$ be the (potential) cost to the government when one is not given Medicaid eligibility. Note that $C(0) = 0$ by construction. Even when given eligibility, one might not enroll and thus incur zero cost to the government. So given the current expenditure is \$6,000 per *enrollee*, the implied expenditure *per capita under eligibility policy* $g(X)$ is actually $\$6,000 \cdot E\left[M(1)g(X)\right]$. The reason is that the expected enrollment rate is only $Pr\{M(1) = 1 \mid X \in g\}$. So the per capita *excess* cost of Medicaid eligibility policy $g(X)$ relative to the current level is

$$R = C(1) - \$6,000 \cdot M(1).$$

We only observe actual cost ($C_i = D_i C_i(1) + (1 - D_i)C_i(0)$) and actual Medicaid enrollment (M_i) in OHIE. We calculate $Z_i = C_i - \$6,000 \cdot M_i$.

Assumption 8. Suppose for all $P \in \mathcal{P}$, the following statements hold for the OHIE:

Independent characteristics: $Pr\{D_i = 1 \mid V_i, X_i\} = Pr\{D_i = 1 \mid V_i\}$

Unconfoundedness: $(Y(1), Y(0), C(1), M(1)) \perp D_i | V_i$.

Bounded attributes: the support of variables X_i, Y_i and Z_i are bounded.

Strict overlap: There exist $\kappa \in (0, 1/2)$ such that the propensity score satisfies $p(v) \in [\kappa, 1 - \kappa]$ for all $v \in \mathcal{V}$.

D.3.1 Uniform convergence of $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$

We want to show the recentered empirical processes $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ converge to mean-zero Gaussian processes G_P^W and G_P^B with covariance functions $\Sigma_P^W(\cdot, \cdot)$ and $\Sigma_P^B(\cdot, \cdot)$ respectively uniformly over $P \in \mathcal{P}$. The covariance functions are uniformly bounded, with diagonal entries bounded away from zero uniformly over $g \in \mathcal{G}$. Take $\widehat{W}_n(\cdot)$ for example, the recentered empirical processes is

$$\sqrt{n}\left(\frac{1}{n}\sum_i \Gamma_i^* \cdot g(X_i) - E_P[\Gamma \cdot g(X_i)]\right)$$

and can be expanded into two terms

$$\sqrt{n}\left(\frac{1}{n}\sum_i \Gamma_i^* \cdot g(X_i) - E_P[\Gamma \cdot g(X_i)]\right) = \frac{1}{\sqrt{n}}\sum_i (\Gamma_i^* - \widetilde{\Gamma}_i) \cdot g(X_i) + \sqrt{n}\left(\frac{1}{n}\sum_i \widetilde{\Gamma}_i \cdot g(X_i) - E_P[\Gamma \cdot g(X_i)]\right). \tag{35}$$

Here $\widetilde{\Gamma}_i$ are the theoretical analogs

$$\widetilde{\Gamma}_i = \gamma^Y(V_i, 1) - \gamma^Y(V_i, 0) + \alpha(V_i, D_i) \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)$$

which is doubly-robust score with the theoretical propensity score and the CEF. A similar expansion holds for $\widehat{B}_n(\cdot)$ involving the theoretical analog

$$\widetilde{R}_i = \gamma^Z(V_i, 1) + \frac{D_i}{p(V_i)} \cdot \left(Z_i - \gamma^Z(V_i, D_i)\right).$$

The following lemmas prove the uniform convergence of $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$.

The last part of the assumption is that we have a uniformly consistent estimate for the covariance function. I argue the sample analog

$$\widehat{\Sigma}^B(g, g') = \frac{1}{n}\sum_i (\widetilde{R}_i)^2 \cdot g(X_i) \cdot g'(X_i) - \left(\frac{1}{n}\sum_i \widetilde{R}_i \cdot g(X_i)\right) \cdot \left(\frac{1}{n}\sum_i \widetilde{R}_i \cdot g'(X_i)\right)$$

is a pointwise consistent estimate for $\widehat{\Sigma}^B(g, g')$. To see this, note that the covariance function $\Sigma_P^B(\cdot, \cdot)$ maps $\mathcal{G} \times \mathcal{G}$ to $\mathbb{R}$. Under Assumption 8 that $\mathcal{G}$ is a VC-class, the product $\mathcal{G} \times \mathcal{G}$ is also a VC-class (see e.g. Lemma 2.6.17 of van der Vaart and Wellner (1996)). By a similar argument leading to Lemma 4 and 6 , we conclude the uniform consistency of $\widehat{\Sigma}^B(\cdot, \cdot)$.

Lemma 2. Let $\mathcal{G}$ be a VC-class of subsets of $\mathcal{X}$ with VC-dimension $v < \infty$. The following sets of functions from $\mathcal{A}$ to $\mathbb{R}$

$$\mathcal{F}^W = \{\widetilde{\Gamma}_i \cdot g(X_i) : g \in \mathcal{G}\}$$
$$\mathcal{F}^B = \{\widetilde{R}_i \cdot g(X_i) : g \in \mathcal{G}\}$$

are VC-subgraph class of functions with VC-dimension less than or equal to v.

Lemma 3. For all P in the family $\mathcal{P}$ of distributions satisfying Assumption 8, for all $g \in \mathcal{G}$ we have

$$E_P[\widetilde{\Gamma}_i \cdot g(X_i)] = W(g; P)$$
$$E_P[\widetilde{R}_i \cdot g(X_i)] = B(g; P)$$

Lemma 4. Let $\mathcal{G}$ satisfy Assumption 8 (VC). Let U_i be a mean-zero bounded random vector of fixed dimension i.e. there exists $M < \infty$ such that i.e. $U_i \in [-M/2, M/2]$ almost surely under all $P \in \mathcal{P}$. Then the uniform deviation of the sample average of vanishes

$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_i U_i \cdot g(X_i) \right| \to_{a.s.} \mathbf{0}.$$

Lemma 5. Let $\mathcal{P}$ be a family of distributions satisfying Assumption 8. Let $\mathcal{G}$ satisfy Assumption 8 (VC). Then $\mathcal{F}^W$ and $\mathcal{F}^B$ are P-Donsker for all $P \in \mathcal{P}$. That is, the empirical process indexed by $g \in \mathcal{G}$

$$\sqrt{n} \cdot (\frac{1}{n} \sum_i \widetilde{\Gamma}_i \cdot g(X_i) - W(g; P))$$

converge to a Gaussian process $\mathcal{GP}(0, \Sigma_P^W(\cdot, \cdot))$ uniformly in $P \in \mathcal{P}$, and the empirical process indexed by $g \in \mathcal{G}$

$$\sqrt{n} \cdot (\frac{1}{n} \sum_i \widetilde{R}_i \cdot g(X_i) - B(g; P))$$

converge to a Gaussian process $\mathcal{GP}(0, \Sigma_P^B(\cdot, \cdot))$ uniformly in $P \in \mathcal{P}$,

Lemma 6. Let $\mathcal{P}$ be a family of distributions satisfying Assumption 8. Let $\mathcal{G}$ satisfy Assumption 8 (VC). Then the estimation errors vanishes

$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \frac{1}{\sqrt{n}} \sum_i (\Gamma_i^* - \widetilde{\Gamma}_i) \cdot g(X_i) \right| \to_p 0$$
$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \frac{1}{\sqrt{n}} \sum_i (R_i^* - \widetilde{R}_i) \cdot g(X_i) \right| \to_p 0$$

D.3.2 Proofs of auxiliary lemmas

Proof. **Proof of Lemma 2.** This lemma follows directly from Lemma A.1 of Kitagawa and Tetenov (2018). □

Proof. **Proof of Lemma 3.** Under Assumption 8, we prove each $\widetilde{\Gamma}_i$ is an conditionally unbiased estimate for Γ:

$$E_P[\widetilde{\Gamma}_i g(X_i)] = E_P[E_P[\widetilde{\Gamma}_i \mid X_i] g(X_i)] = E_P[E_P[\Gamma \mid X_i] g(X_i)] = W(g; P).$$

I focus on $E_P[\widetilde{\Gamma}_i \mid X_i] = E_P[E_P[\widetilde{\Gamma}_i \mid V_i, X_i] \mid X_i]$. However, conditional on V_i, by unconfoundedness and strict overlap we have

$$\begin{aligned} E_P[\widetilde{\Gamma}_i \mid V_i, X_i] &= E[Y_i \mid V_i, X_i, D_i = 1] - E[Y_i \mid V_i, X_i, D_i = 0] \\ &= E[Y_i(1) - Y_i(0) \mid V_i, X_i] = E[\Gamma_i \mid V_i, X_i] \end{aligned}$$

Specifically

$$\begin{aligned} E[\widetilde{\Gamma}_i \mid V_i, X_i] &= E\left[\gamma^Y(V_i, 1) - \gamma^Y(V_i, 0) \mid V_i, X_i\right] + E\left[\alpha(V_i, D_i) \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \mid V_i, X_i\right] \\ &= E\left[Y_i \mid V_i, D_i = 1\right] - E\left[Y_i \mid V_i, D_i = 0\right] + \\ &\quad E\left[Y_i(1) - Y_i(0) \mid V_i, X_i\right] - \left(E\left[Y_i \mid V_i, D_i = 1\right] - E\left[Y_i \mid V_i, D_i = 0\right]\right) \\ &= E[\Gamma \mid V_i, X_i] \end{aligned}$$

Specifically, we expand $E[\alpha(V_i, D_i)Y_i \mid V_i, X_i]$

$$
\begin{aligned}
&= E\left[\frac{Y_i}{p(V_i)} \mid D_i = 1, V_i, X_i\right] \cdot Pr\{D_i = 1 \mid V_i, X_i\} - E\left[\frac{Y_i}{1-p(V_i)} \mid D_i = 0, X_i\right] \cdot Pr\{D_i = 0 \mid V_i, X_i\} \\
&= E\left[D_iY_i(1) + (1-D_i)Y_i(0) \mid D_i = 1, V_i, X_i\right] - E\left[D_iY_i(1) + (1-D_i)Y_i(0) \mid D_i = 0, V_i, X_i\right] \\
&= E\left[Y_i(1) \mid V_i, X_i\right] - E\left[Y_i(0) \mid V_i, X_i\right]
\end{aligned}
$$

where the second line holds by independent characteristic such that

$$
Pr\{D_i = 1 \mid V_i, X_i\} = Pr\{D_i = 1 \mid V_i\} =: p(V_i)
$$

and the last line follows from unconfoundedness. Similarly, we show

$$
E\left[\alpha(V_i, D_i) \cdot \gamma^Y(V_i, D_i) \mid V_i, X_i\right] = E\left[Y_i \mid V_i, D_i = 1\right] - E\left[Y_i \mid V_i, D_i = 0\right]
$$

Similar argument holds for $E_P[\widetilde{R}_i \cdot g(X_i)]$, with the only modification:

$$
\begin{aligned}
&E[\frac{D_i}{p(V_i)}C_i \mid V_i, X_i] = E\left[\frac{C_i}{p(V_i)} \mid D_i = 1, V_i, X_i\right] \cdot Pr\{D_i = 1 \mid V_i, X_i\} \\
&= E[C_i(1) \mid V_i, X_i]
\end{aligned}
$$

□

Proof. **Proof of Lemma 4.** Denote the following set of functions from $\mathcal{U}$ to $\mathbb{R}$

$$
\mathcal{F}^U = \{U_i \cdot g(X_i) : g \in \mathcal{G}\}
$$

and it has uniform envelope $\bar{F} = M/2$ since U_i is bounded. This envelop function is bounded uniformly over $\mathcal{P}$. Also, by Assumption 8 (VC) and Lemma 2, $\mathcal{F}^U$ is VC-subgraph class of functions with VC-dimension at most v. It satisfies the uniform entropy condition for any given P by Theorem 2.6.7 of van der Vaart and Wellner (1996). Since the bound on entropy only depends on v, the uniform entropy condition is satisfied for every $P \in \mathcal{P}$. By Theorem 2.8.1 of van der Vaart and Wellner (1996), we conclude that $\mathcal{F}^U$ is P-Glivenko–Cantelli uniformly in $P \in \mathcal{P}$. □

Proof. **Proof of Lemma 5.** Note that Assumption 8 imply that $\mathcal{F}^W$ and $\mathcal{F}^B$ have uniform envelope $\bar{F} = M/(2\kappa)$. $\mathcal{F}^W$ and $\mathcal{F}^B$ thus have square integrable envelop functions uniformly over $\mathcal{P}$. Also, by Assumption 8 (VC) and Lemma 2, $\mathcal{F}^W$ and $\mathcal{F}^B$ are VC-subgraph class of functions with VC-dimension at most v. Therefore they satisfy the uniform entropy condition for any given P by Theorem 2.6.7 of van der Vaart and Wellner (1996). Since the bound on entropy only depends on v, the uniform entropy condition is satisfied for every $P \in \mathcal{P}$. By Theorem 2.8.3 of van der Vaart and Wellner (1996), we conclude that $\mathcal{F}^W$ and $\mathcal{F}^B$ are P-Donsker uniformly in $P \in \mathcal{P}$. □

Proof. **Proof of Lemma 6.** I focus on the deviation in $\Gamma_i^* - \widetilde{\Gamma}_i$. The deviation in $R_i^* - \widetilde{R}_i$ can be proven to vanish in a similar manner. Denote $\Delta\gamma^Y(V_i) = \gamma^Y(V_i, 1) - \gamma^Y(V_i, 0)$. For any fixed policy g, we expand the deviation into

$$
\begin{aligned}
\frac{1}{\sqrt{n}}\sum_i(\Gamma_i^* - \widetilde{\Gamma}_i)g(X_i) =& \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right) \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \qquad (36) \\
&+ \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\Delta\widehat{\gamma}^Y(V_i) - \Delta\gamma^Y(V_i) - \alpha(V_i, D_i) \cdot \left(\widehat{\gamma}^Y(V_i, D_i) - \gamma^Y(V_i, D_i)\right)\right) \\
&- \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\gamma}^Y(V_i, D_i) - \gamma^Y(V_i, D_i)\right) \cdot \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right) \cdot
\end{aligned}
$$

Denote these three summands by $D_1(g)$, $D_2(g)$ and $D_3(g)$. I will bound all three summands separately. Recall we use the full sample to estimate the propensity score and the CEF with a saturated model. The purpose of the above expansion is to separately bound the estimation error from the estimated CEF and propensity score, and the deviation from taking sample averages. For cross-fitted estimators for the propensity score and the CEF, a similar bound can be found in Athey and Wager (2020).

Uniform consistency of the estimated CEF and propensity score Denote with $b(V_i, D_i)$ the dictionary that spans (V_i, D_i), and $b(V_i,)$ the dictionary that spans V_i. The saturated models are therefore parameterized as $\gamma^Z(V_i, D_i) = \gamma' b(V_i, D_i)$ and $p(V_i) = \beta' b(V_i)$. Under standard argument, the OLS estimators $\widehat{\gamma}$ and $\widehat{\beta}$ are asymptotically normal uniformly over $P \in \mathcal{P}$:

$$\sqrt{n} \cdot (\widehat{\gamma} - \gamma) = O_P(1), \ \sqrt{n} \cdot \left(\widehat{\beta} - \beta\right) = O_P(1).$$

Furthermore, the in-sample L_2 errors from the estimated CEF and propensity score vanish. Consider

$$\frac{1}{n} \sum_i \left(\widehat{\gamma}^Z(V_i, D_i) - \gamma^Z(V_i, D_i)\right)^2 = \frac{1}{n} \sum_i \left((\widehat{\gamma} - \gamma)' b(V_i, D_i)\right)^2 = (\widehat{\gamma} - \gamma)' \widehat{M} (\widehat{\gamma} - \gamma)$$

where $\widehat{M} = \frac{1}{n} \sum_i b(V_i, D_i) b(V_i, D_i)'$. It converges in probability to a fixed matrix $M = E[b(V_i, D_i) b(V_i, D_i)']$. So the in-sample L_2 error from the estimated CEF vanishes at the rate of $n^{-1/2}$.

Similarly, consider

$$\frac{1}{n} \sum_i \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right)^2 = \frac{1}{n} \sum_i \left(\frac{1}{\widehat{\beta}' b(V_i)} - \frac{1}{\beta' b(V_i)}\right)^2 D_i^2 + \left(\frac{1}{1 - \widehat{\beta}' b(V_i)} - \frac{1}{1 - \beta' b(V_i)}\right)^2 (1 - D_i)^2$$

With a first-order Taylor approximation, for each term in the summand, the dominating term would be

$$\frac{1}{n} \sum_i \left(\left(\widehat{\beta} - \beta\right)' \frac{-b(V_i)}{(\beta' b(V_i))^2}\right)^2 D_i^2 = \left(\widehat{\beta} - \beta\right)' \left(\frac{1}{n} \sum_i \frac{b(V_i) b(V_i)'}{(\beta' b(V_i))^2} D_i^2\right) \left(\widehat{\beta} - \beta\right)$$

where the middle term converges to a fixed matrix as implied by $\beta' b(V_i)$ being bounded away from zero and one. So the in-sample L_2 error from the estimated propensity score also vanishes at the rate of $n^{-1/2}$.

Bounding the deviation I now bound each term in (36). Plugging in the first-order Taylor approximation with a remainder term to the estimated propensity score, we have

$$\begin{aligned}
D_1(g) =& \frac{1}{\sqrt{n}} \sum_i g(X_i) \left(\left(\frac{1}{\widehat{\beta}' b(V_i)} - \frac{1}{\beta' b(V_i)}\right) D_i + \left(\frac{1}{1 - \widehat{\beta}' b(V_i)} - \frac{1}{1 - \beta' b(V_i)}\right)(1 - D_i)\right) \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \\
=& \frac{1}{\sqrt{n}} \sum_i g(X_i) \left(\widehat{\beta} - \beta\right)' \frac{-b(V_i)}{(\beta' b(V_i))^2} \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) + \\
& \frac{1}{\sqrt{n}} \sum_i g(X_i) \left(\widehat{\beta} - \beta\right)' \frac{b(V_i) b(V_i)'}{(\widetilde{\beta}' V_i)^3} \left(\widehat{\beta} - \beta\right) \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \\
=& \underbrace{\sqrt{n} \left(\widehat{\beta} - \beta\right)'}_{O_P(1)} \underbrace{\frac{1}{n} \sum_i g(X_i) \frac{-b(V_i)}{(\beta' b(V_i))^2} \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)}_{o_P(1)} + o_P(1)
\end{aligned}$$

where $\widetilde{\beta}$ is a sequence between $\widehat{\beta}$ and β. This remainder term therefore converges to zero. Uniform convergence of the sample average follows from Lemma 4: the random vector $\frac{-b(V_i)}{(\beta' b(V_i))^2} \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)$ is mean-zero, has fixed dimension, and bounded.

I can decompose $D_2(g)$ into the product of two terms

$$D_2(g) = \underbrace{\sqrt{n} \left(\widehat{\gamma} - \gamma\right)'}_{O_p(1)} \frac{1}{n} \sum_i g(X_i) \left(\Delta b(V_i, D_i) - \alpha(V_i, D_i) \cdot b(V_i, D_i)\right)$$

Uniform convergence of the sample average again follows from Lemma 4. I thus conclude $D_1(g)$ and $D_2(g)$ vanish uniformly over $g \in \mathcal{G}$ and over $P \in \mathcal{P}$.

For $D_3(g)$, we apply the Cauchy-Schwarz inequality to note that

$$D_3(g) \leq \sqrt{n} \cdot \sqrt{\frac{1}{n} \sum_i \left(\widehat{\gamma}^Y(V_i, D_i) - \gamma^Y(V_i, D_i)\right)^2} \cdot \sqrt{\frac{1}{n} \sum_i \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right)^2}$$

The terms in the square root are the in-sample L_2 errors from the estimated CEF and propensity score, which vanish at the rate of $n^{-1/2}$ uniformly over $P \in \mathcal{P}$ as shown in the paragraph above. I thus conclude $D_3(g)$ vanishes uniformly over $g \in \mathcal{G}$ and over $P \in \mathcal{P}$. $\square$